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The Existence and Nonexistence of Critical Points in Bounded Flows*

G. STEPHEN JONES AND JAMES A. YORKE

Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742

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1. INTRODUCTION

Consider a continuous function $g : R^n \to R^n$ and a differential equation of the form

$$\dot{x} = g(x), \tag{1}$$

whose solutions are uniquely determined by their initial values. A problem of considerable importance to the qualitative theory of differential equations concerns the specification of general conditions under which such an autonomous system necessarily has a critical point. For $R^n = R^2$ it is well known from Poincaré–Bendixson theory that if there exists a positively or negatively bounded solution, then the system has at least one critical point. It is natural, therefore, to ask if the same is true for systems of high dimensions.

In Section 2 of this paper we shall show by examples that in Euclidean spaces of dimensions larger than 2 that differential equations can have bounded solutions and no critical points. Of more significance the second example presented shows that in $R^n, n \geq 3$, all solutions of a differential equation may be bounded without necessitating the existence of a critical point. The referee has brought to our attention that a previous example exhibiting such behavior is described in the unpublished lecture notes of L. Markus (attributed to Terasaka). Our construction, however, is simpler, and we have exhibited explicitly the corresponding differential equation. In Section 3 we prove a theorem specifying conditions which assure the existance of critical points for autonomous differential equations acting in any space R^n.

For simplicity in presenting our results on the existence of critical points

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let us assume that Equation (1) has solutions defined for all time passing through each point in R^n which are uniquely determined and continuously dependent on their initial data. We shall denote by $x(t; \xi)$ the solution passing through the point ξ at time $t = 0$. If $S \subset R^n$ and t_1 is a fixed time, then $x(t_1 ; S)$ denotes the set $\{x(t_1 ; \xi) : \xi \in S\}$. A set S is said to be *constrained* under the action of the differential equation (1) if there exist $t_1 \neq 0$ such that $x(t_1 ; \bar{S}) \subset S$. Let S be a set whose complement in R^n has at most one unbounded component. Let S^* denote the union of S with all bounded components of its complement. If S^* is mapped onto a convex set under a homeomorphism of R^n onto itself, then we shall say that S is *homeomorphically externally convex*. We shall prove the following theorem on the existence of critical points.

THEOREM 1. *If Equation (1) constrains a bounded open set S which is homeomorphically externally convex, then Equation (1) has a critical point in S^*.*

The second example in Section 2 is constructed by partitioning and filling R^n with flows on tori. This theorem suggests that in some sense such constructions characterize bounded flows without critical points. That is, our results come close to saying that critical point free bounded flows necessarily partition R^n into torus resembling structures or their more complicated analogues in higher dimensions. These ideas hint at many intriguing problems concerning the geometry of flows.

We note that if equation (1) constrains a bounded open set S which is homeomorphically externally convex, then since for any fixed t_1, $x(t_1 ; \cdot)$ is a homeomorphism on R^n, it also constrains S^*. We also note that it follows from the fact that $x(t_1 ; \cdot)$ is a homeomorphism that $x(t_1 ; S) \supset S$ and $x(t_1 ; \partial S^*) \cap \partial S^*$ being empty (where ∂S^* denotes the boundary of S^*) imply equation (1) constrains $x(t_1 ; S)$. Hence $x(t_1 ; S) \supset S$ and $x(t_1 ; \partial S^*) \cap \partial S^*$ empty imply the existence of a critical point.

Theorem 1 in terms of dynamical systems $\pi : R \times R^n \to R^n$ takes the following form.

THEOREM 1'. *Let $\pi : R \times R^n \to R^n$ be a dynamical system which constrains a bounded open set S which is homeomorphically externally convex. Then π has a critical point in S^*.*

In addition to Theorems 1 and 1' we shall prove the following corollary which extends results such as contained in Bhatia and Szego [*1*].

COROLLARY 1. *Let a dynamical system $\pi : R \times R^n \to R^n$ have a compact weak attractor W. Suppose U is an open neighborhood of W and $\pi(t; U)$ for all t sufficiently large is contained in a compact homeomorphically externally* *conves subset of the region of weak attraction of W. Then π has at least one critical point.*

2. Examples of Bounded Flows Having no Critical Points

Construction of flows on R^3 which have some bounded solutions but no critical points can be done very simply as illustrated in Figure 1. A differential equation generating such a flow is given in the next paragraph.

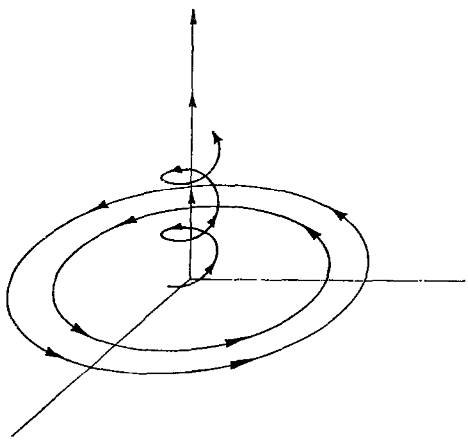

Fig. 1

Let $x = (x_1, x_2, x_3)$ and let the continuous function $\rho : R \to R$ be such that $\rho(0) \neq 0$ and $\rho(r) = 0$ for $r \geqslant 1$. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \\ \dot{x}_3 &= \rho(x_1^2 + x_2^2) \end{aligned} \tag{2}$$

Then for any $t_0 \in R$ and $\alpha \geqslant 1$ and $\beta \in R$, $(\alpha \cos(t + t_0), \alpha \sin(t + t_0), \beta)$ is a bounded (periodic) solution of (2). Clearly $(x_2)^2 + (-x_1)^2 + \rho(x_1^2 + x_2^2)^2 > 0$ for all $(x_1, x_2, x_3) \in R^3$, so there are no critical points.

Let us now illustrate a flow in which all trajectories are bounded but which contains no critical point. In Figure 2 we indicate a flow on a torus imbedded in a larger flow, shown for a segment of a larger torus. The flow on the surface of both the small and the large torus is invariant and parallel, but internal to the large torus and engulfing the smaller torus the flow assumes a whirlpool character which allows continuity to the maintained.

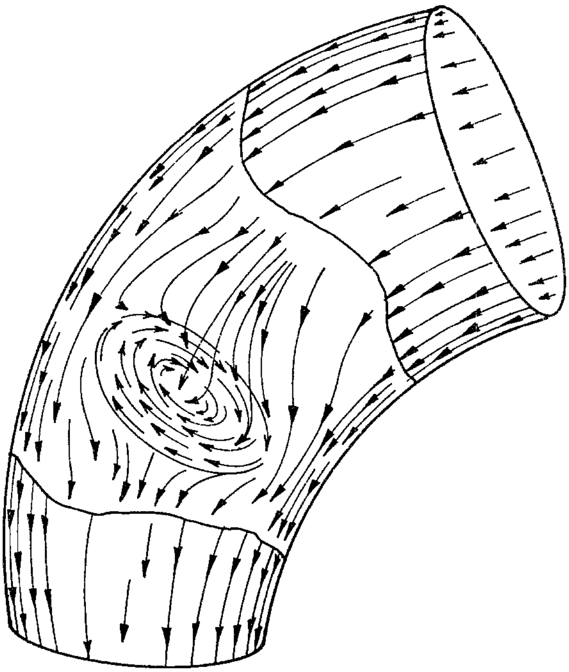

FIG. 2

Now if we think of the large torus as embedded in a still larger torus with precisely the same whirlpool type flow about it as above the small torus contained in its interior, then we see how a larger portion of R^3 can be filled with a flow which has no critical points. Obviously we can continue this imbedding process indefinitely and thereby fill R^3 with bounded flows without introducing a critical point. The flow in the smallest torus is the natural periodic flow.

We now present a differential equation generating precisely such a flow.

Let $c_r = \frac{2}{3}(4^{r+1} - 4)$, $\tilde{c}_r = (0, c_r, 0)$ and $i(r) = 2 - (-1)^r$ for $r = 0, 1, 2,\ldots$. Let C^r be the circle in R^3 with center $\tilde{c}_r$ and radius $4^r \cdot 2$ which lies in the plane determined by v_2 and $v_{i(r)}$, where $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$. For $r = 0, 1,\ldots$, let T^r be the solid torus

$$\{x \in R^3 : d(x, C^r) \leqslant 4^r\}.$$

Note that the center $\tilde{c}_r$ of T^r is on the circle C^{r+1}, so $T^r \subset T^{r+1}$ and $\bigcup^{\infty} T^r = R^3$. Define the linear functions

$$G_3(x) = (0, -x_3, x_2)$$

$$G_1(x) = (x_2, -x_1, 0)$$

If $x \in T^r$, then $G_{i(r)}(x - \tilde{c}) \neq 0$ since $G_{i(r)}(y - \tilde{c}) = 0$ only for y on the central axis of T^r, that is, on the line through $\tilde{c}$ perpendicular to the $v_2 v_{i(r)}$ plane. Define $h_0(x) = \min\{1, 1 - d(x, T^0)\}$. For $r = 1, 2, \ldots,$ define $h_r(x) = \min\{1, 1 - d(x, T^r), d(x, T^{r-1})\}$. The differential equation in R^3 is

$$\dot{x} = \sum_{r=0}^{\infty} G_{i(r)}(x - \tilde{c}_r)\, h_r(x) \tag{3}$$

We shall write $g_r(x)$ for $G_{i(r)}(x - \tilde{c}_r)\, h_r(x)$ and $g(x) = \sum_{r=0}^{\infty} g_r(x)$ so (3) is $\dot{x} = g(x)$. By definition the functions h_r and $G_{i(r)}$ are Lipschitzian with constant 1, but the finite sums and products of Lipschitzian functions are locally Lipschitzian. In particular if f_1 and f_2 are functions with Lipschitz constants L_1 and L_2 on a compact set Q, i.e., $|f_i(x_1) - f_i(x_2)| \leqslant L_i \mid x_1 - x_2 \mid$ for $i = 1, 2$ and for $x_1, x_2 \in Q$, then $f_1 f_2$ is Lipschitz on Q with Lipschitz constant $L_1 \sup_{x \in Q} |f_2(x)| + L_2 \sup_{x \in Q} |f_1(x)|$. Since $R^3 = \bigcup_{r=0}^{\infty} T^r$ and $T^r \subset T^{r+1}$, and $h_m(x) = g_m(x) = 0$ for $x \in T^r$ and $m > r$, for any T^r, g is the finite sum $g_0 + \cdots + g_r$. Therefore g is (locally) Lipschitzian on each T^r and so is locally Lipschitzian on R^3, and solutions of (3) are unique.

We now show $g(x) \neq 0$ for all $x \in R^n$. Fix x. Let $r = \min\{s : x \in T^s\}$. Since $h_m(x) = 0$ for $m > r$, if $r = 0$, $g(x) = g_0(x) = G_1(x - 0) \neq 0$. Suppose $r > 0$. To fix $i(r)$, let r be even so $i(r) = 1$. The argument for n odd is similar. If $d(x, T^m) \geqslant 1 (m < r)$ then $h_m(x) = 0$. If $d(x, T^{r-1}) \geqslant 1$, $h_r(x) = 1$ and $g(x) = g_r(x) = G_1(x - \tilde{c}_r) \neq 0$. If $0 < d(x, T^{r-1}) < 1$, then $h_m(x) = 0$ unless $m = r$ or $r - 1$. The first component of $G_{i(r-1)}$ (that is, of G_3) is zero and hence of $g_{r-1}(x)$ is 0. Therefore, the first component of $g(x)$ is the first component of $g_r(x)$, which is $(x_2 - c_r)\, h_r(x)$. To show that $g(x) \neq 0$, then, it suffices that $x_2 \neq c_r$, (which we will see holds because $| x_2 - c_{r-1} |$ is small). Since $\tilde{c}_{r-1}$ is the center of C_{r-1}, if $y \in T^{r-1}$, then $d(y, \tilde{c}_{r-1}) \leqslant \text{radius } C^{r-1} + d(y, C_{r-1}) \leqslant 2 \cdot 4^{r-1} + 4^{r-1}$, and $| x_2 - c_{r-1} | \leqslant |(x_1, x_2, x_3) - (0, c_{r-1}, 0)| = d(x, \tilde{c}_{r-1}) < 3 \cdot 4^{r-1} + 1$. The triangle inequality gives

$$\begin{aligned} | x_2 - c_r | &\geqslant | c_r - c_{r-1} | - | x_2 - c_{r-1} | \\ &\geqslant (2 \cdot 4^r - 2 \cdot 4^{r-1}) - (3 \cdot 4^{r-1} + 1) > 0 \qquad \text{for} \quad r \geqslant 1. \end{aligned}$$

Hence $g(x) \neq 0$.

The boundedness of all solutions follows from the fact that $R^3 = \bigcup_{r=0}^{\infty} T^r$, $T^r \subset T^{r+1}$, and each T^r is invariant. To see that each T^r is invariant, note that for any x in the boundary of T^r, $g(x) = g_r(x)$ and the trajectory through x is a circle remaining on the boundary of T^r. Clearly all solutions of (3) are bounded but there are no constant solutions.

We can trivially extend our example to flows of dimension $n > 3$ by considering $(x, y) \in R^3 \times R^{n-3}$ and the system

$$\dot{x} = g(x)$$
$$\dot{y} = 0$$

3. Critical Points for Differential Equations Constraining Homeomorphically Convex Sets

In this section we shall prove Theorem 1 and relate it to other results. We first note that we could prove this result using asymptotic fixed point theory such as presented in references [*2*]. This would have the advantage that most of our argument would go over directly to discrete semigroups. We shall not use this approach here, however, since for continuous flows it is possible to use simpler tools and keep our presentation more self-contained.

We shall formulate our proof of Theorem 1 through the use of two lemmas which are of interest in themselves. Let us first introduce continuous mappings of the form $\pi : R \times R^n \to R^n$ where $\pi(0; \xi) = \xi$ and $\pi(t_1 + t_2 ; \xi) = \pi(t_1 ; \pi(t_2 ; \xi))$. A mapping such as π is referred to as transformation group or dynamical system. Clearly a solution x of Equation (1) is a dynamical system.

Lemma 1. *Let S be a bounded open convex set and let $\pi : R \times R^n \to R^n$ be a dynamical system. If $\pi(t; \bar{S}) \subset S$ for all t sufficiently large, then π has a critical point in S. That is, there exists ξ^* in S such that $\pi(t; \xi^*) = \xi^*$ for all $t \in R$.*

Proof. By hypothesis there exists $t^* > 0$ such that for all $t \geqslant t^*$, $\pi(t, \bar{S}) \subset S$. From the Brouwer fixed point theory we may conclude that for all $t \geqslant t^*$ there exists $\xi_t \in S$ such that $\pi(t; \xi_t) = \xi_t$. By the standard homotopy argument (see [*2*], page 49) either there exists ξ_t such that $\pi(t; \xi_t) = \xi_t$ for all $t \in [0, t^*)$ or there exists $t_1 \in (0, t^*)$ and $\xi_{t_1} \in \bar{S} \setminus S$ such that $\pi(f_1 ; \xi_{t_1}) = \xi_{t_1}$. But the existence of $\xi_{t_1} \in \bar{S} \setminus S$ implies $\pi(kt_1 ; \xi_{t_1}) \subset \bar{S} \setminus S$ for all integers $k \geqslant 1$ and this is contrary to our hypothesis that $\pi(t; \bar{S})$ is eventually contained in S. Hence we can only conclude that for all $t \in (0, \infty)$ there exists $\xi_t \in S$ such that $\pi(t; \xi_t) = \xi_t$.

Let $\{t_k\} \to 0$ and let $\{\xi_{t_k}\}$ denote a corresponding sequence of points in S such that $\pi(t_k ; \xi_{t_k}) = \xi_{t_k}$. $\{\xi_{t_k}\}$ must have a limit point which we denote by ξ^* contained in $\bar{S}$. We shall show that ξ^* is a critical point of π.

If ξ^* is not a critical point we can choose $h^* > 0$ such that $\pi(h^*; \xi^*) \neq \xi^*$. We can thus choose a closed neighborhood $N(\pi(h^*; \xi^*))$ of $\pi(h^*; \xi^*)$ not containing ξ^*. By continuity we can choose a neighborhood $N_0(\xi^*)$ of ξ^* with $N_0(\xi^*) \cap N(\pi(h^*; \xi^*))$ empty and $\epsilon > 0$ such that $\xi \in N_0(\xi^*)$ and $| h - h^* | < \epsilon$ imply $\pi(h; \xi) \in N(\pi(h^*; \xi^*))$. We can choose $t_k \in \{t_k\}$ such that $\xi_{t_k} \in N_0(\xi^*)$ and for some positive integer n, $| nt_k - h^* | < \epsilon$. But $\xi_{t_k} = \pi(nt_k ; \xi_{t_k})$ and we have that $\pi(nt_k ; \xi_{t_k}) \in N(\pi(h^*, \xi^*))$. Since $N_0(\xi^*) \cap N(\pi(h^*; \xi^*))$ is empty we clearly have a contradiction. It follows that ξ^* must be a critical point of π and our proof is complete.

LEMMA 2. *Let $S \subset R^n$ be any bounded open set and let S^* be the union of S with all bounded components of its complement in R^n. If for some $t_1 > 0$, $\pi(t_1 ; \bar{S}) \subset S$, then $\pi(t; \bar{S})$ and $\pi(t; \bar{S}^*)$ for all t sufficiently large are contained in S and S^* respectively.*

Proof. $\pi(t_1 , \bar{S}) \subset S$ implies the existence of a compact set $U_0 \subset S$ such that $\pi(t; \bar{S}) \subset U_0$ for all t in some interval $[t_1 - 2\delta_0 , t_1 + 2\delta_0]$, $\delta_0 > 0$. Furthermore, we may choose a compact set $U_1 \subset S$ and $\delta_1 > 0$ such that $\pi(t; U_0) \subset U_1$ for all $t \in [0, \delta_1]$. Letting $h = \min\{\delta_0 , \delta_1\}$ there exists an integer m such that $\pi(mh; \bar{S})$ and $\pi((m + 1)h; \bar{S})$ are contained in U_0. It follows that for all integers k_0 and k_1, that $\pi((k_0 m + k_1(m + 1))h; \bar{S})$ is contained in U_0. But every integer greater than $(m - 1)m$ is expressible in the form $k_0 m + k_1(m + 1)$, so for all $n \geqslant (m - 1)m$ we have $\pi(nh; \bar{S}) \subset U_0$. This, of course, implies $\pi(t; \bar{S}) \subset U_1 \subset S$ for all $t \geqslant (m - 1)mh$, and the first part of Lemma 2 is proved.

Let C_∞ be the unbounded component of $R^n \setminus S$. Assume $S \neq S^*$ and let C be any bounded component of $R^n \setminus S$. As a property of homeomorphic images we have that for every t in R, $\pi(t; C)$ must be a bounded component of $\pi(t; R^n \setminus S)$ which is separated from $x(t; C_\infty)$ by $x(t; S)$. For $t \geqslant (m - 1)mh$ this clearly implies $\pi(t : C) \subset S^*$. Since C is an arbitrary bounded component of $R^n \setminus S$, it follows that $\pi(t : S^* \setminus S) \subset S^*$ and consequently

$$\pi(t; (S^* \setminus S) \cup \bar{S}) = x(t; \bar{S}^*) \subset S^* \qquad \text{for all} \quad t \geqslant (m - 1)mh.$$

Thus our proof of Lemma 2 is complete.

THEOREM 1. *If Equation* (1) *constrains a bounded open set S which is homeomorphically externally convex, then Equation* (1) *has a critical point contained in S^*.*

Proof. By hypothesis there exists a homeomorphism $\phi : R^n \to R^n$ such that $\phi(S^*)$ is a bounded open convex set. Letting $x(\cdot; \xi)$ denote a solution of equation (1) we define $\pi : R \times R^n \to R^n$ by the formula

$$\pi(t; \xi) = \phi(x(t; \phi^{-1}(\xi)))$$

and note that it is a dynamical system. By hypothesis there exist $t_1 \neq 0$ such that $x(t_1 ; \bar{S}) \subset S$. Without loss of generality we can assume $t_1 > 0$.

By Lemma 2 it follows that $x(t; \bar{S}) \subset S$ for all t greater than some $t_2 > 0$. Clearly then $\pi(t; \phi(\bar{S}^*)) \subset \phi(S^*)$ for all $t \geqslant t_2$. By Lemma 1 there exists $\xi^* \in \phi(S^*)$ such that $\pi(t; \xi^*) = \xi^*$ for all $t \in R$. Obviously then $x(t; \phi^{-1}(\xi^*)) = \phi^{-1}(\xi^*)$ for all $t \in R$ and the theorem is proved.

It should be clear that Theorem 1 is in no way restricted to dynamical systems generated by differential equations. That is to say by the same proof we have the following theorem:

THEOREM 2. *Let $\pi : R \times R^n \to R^n$ be a dynamical system which constrains a bounded open set S which is homeomorphically externally convex. Then π has a critical point in S^*.*

Let $\pi : R \times R^n \to R^n$ be a dynamical system and let W and G be sets such that $W \subset G \subset R^n$ where W is closed and G is open. Suppose for every point ξ in G there exists a sequence $\{t_n\} \to \infty$ such that $\pi(t_n ; \xi)$ approaches the set W as $n \to \infty$. Then W is called a weak attractor under the action of π having G contained in its region of weak attraction or influence. In the case where W is bounded with G homeomorphic to R^n and maximal with respect to the weak attraction of W, it has been shown by Bhatia and Szegö [*1*] that π has a critical point. The same type of result can be observed to follow from the asymptotic fixed point theory in [*3*] in a much more general setting. We shall now prove a corollary to Theorem 2 which generalizes these earlier results in a way useful in many applications.

COROLLARY 1. *Let a dynamical system $\pi : R \times R^n \to R^n$ have a compact weak attractor W. Suppose U is an open neighborhood of W and $\pi(t; U)$ for all t sufficiently large is contained in a compact homeomorphically externally convex subset V of the region of weak attraction G of W. Then π has at least one critical point.*

Note. Since W is compact it is easily verified that there exists $t_1 > 0$ such that the set $P = \{\pi(t; W) : t \in [0, t_1]$ is a compact invariant subset of G. If G is homeomorphically convex or homeomorphically externally convex with the union of the bounded components of its complement bounded, then clearly the hypotheses of Corollary 1 are satisfied.

Proof of Corollary 1. By hypothesis there exists $t_1 > 0$ such that $\pi(t; U) \subset V$ for all $t \geqslant t_1$. There exists a homeomorphism $\phi : R^n \to R^n$ such that $\phi(V)$ is externally convex and it follows that we may choose an open externally convex neighborhood Q of $\phi(V)$ whose closure is contained in $\phi(G)$. Now defining $S = \phi^{-1}(Q)$ we have an open bounded homeomorphically externally convex set containing V and whose closure is contained in G. Furthermore, the compactness of $\bar{S}$ implies that there exists $t_2 > 0$

such that for every $x \in \bar{S}$ there exists $t \leqslant t_2$ such that $\pi(t; x) \in U$. Hence for all $t \geqslant t_1 + t_2$, $\pi(t; \bar{S}) \subset S$, so S is constrained by π. It follows from Theorem 2 that π has a critical point.

In closing we note that the results presented on the existence of critical points carry over with very little modification to dynamical systems defined on more general linear spaces. The restriction of our discussion to R^n has been largely for the sake of simplicity rather than necessity. For systems considered on more general spaces the reader is referred to [*4*].

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